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Research Report No.EM-49

GRAPHS OF THE FUNCTION $E(N,\delta) = \sum_{n=1}^{N} n^{-1/2} e^{in\delta}$

by

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ABSTRACT

The function $E = C + iS$ of the title arises in diffraction problems involving gratings and in allied antenna problems. We present curves of S vs. C as functions of δ for $N = 1$ to 10 and $N = 10^2, 10^3, 10^4$. It is found that $|E|_{max}$ always occurs for $\delta = 0$, and the curves from $\delta = 0$ to 2π show N-1 loops; for large N, $|E|_{min}$ occurs for $\delta = \pi$. Curves of C and S as functions of δ are also included.

The function

$$(1) \quad E(N,\delta) = \sum_{n=1}^{N} n^{-1/2} e^{in\delta} = C + iS,$$

where C and S are real, is the limiting form of a Schlömilch series which arises in the multiple scattering solution of the grating of parallel cylinders.[1] It also arises in allied antenna problems involving linear arrays of coupled elements. In these problems, δ is the phase difference of neighboring elements and N is the number of neighbors taken into account. The function E, which involves two variables, is essentially the sum analog of the Fresnel integral.

For small values of N we have evaluated E by graphical methods. Figures 1 to 10 are curves of S vs. C for N = 1 to 10; the variable δ ranges from 0° to 180°. These curves if taken from $\delta = 0^\circ$ to 360° have N-1 loops. With increasing N the train of loops forms a "pigtail" spiral, and $|E|_{min}$ tends to occur for $\delta = 180^\circ$. $|E|_{max}$ always occurs for $\delta = 0$.

Figures 11 to 20 are curves of C and S as functions of δ for N = 1 to 10. The curves for N = 1 are simply the cosine and sine curves; the corresponding curves for N > 1 have essentially the same symmetry properties as the cosine and sine.

Figures 21 to 24 are vector diagrams for several constant values of δ and N increasing from 1 to 10. For a fixed large value of N, graphs of this type may have essentially one of three forms as determined by the constant δ. For $\delta = 0$ (or π) the plot is a straight line or, more generally, for $\delta \leq (\phi + \pi/2)/N$, $\phi = \tan^{-1}(S/C)$, the plot is a 'simple arc' and the resultant of the vectors is much larger than unity, as in Figure 21.* For $(\phi + \pi/2)/N < \phi \leq \pi/2$, the plot is a 'spiral' as in Figures 22 and 23.

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[1] V. Twersky, J. Appl. Phys. 23, 1099 (1952). See also paper by S. Karp; in press.

* The transition value of δ is deduced as follows: for the simple arc case to obtain we require that the N-th vector make an angle less than 90° with the resultant, i.e., if the angle is greater than 90° the graph has already begun to spiral; hence, for large N, the transition from simple arc to spiral occurs for $N\delta = \phi + \pi/2$. We are grateful to W. Magnus for suggesting this.

For $\pi/2 < \delta < \pi$, the plot forms a jagged 'star patch' which intersects itself many times and has a resultant less than unity; cf. Fig. 24.

For large values of N, a more rapidly convergent representation of E suitable for numerical calculations may be obtained by means of the 'Euler sum formula'[2]; thus

$$(2) \quad C(N,\delta) = \sqrt{\frac{2\pi}{\delta}}\left[\mathcal{C}(N\delta) - \mathcal{C}(\delta)\right] + \frac{1}{2}\left(\cos\delta + \frac{\cos N\delta}{\sqrt{N}}\right)$$

$$+ \frac{1}{12}\left(\frac{\cos\delta}{2} + \delta\sin\delta - \frac{\cos N\delta}{2\sqrt{N^3}} - \frac{\delta\sin N\delta}{\sqrt{N}}\right) \cdots ,$$

$$S(N,\delta) = \sqrt{\frac{2\pi}{\delta}}\left[\mathcal{S}(N\delta) - \mathcal{S}(\delta)\right] + \frac{1}{2}\left(\sin\delta + \frac{\sin N\delta}{\sqrt{N}}\right)$$

$$+ \frac{1}{12}\left(\frac{\sin\delta}{2} - \delta\cos\delta - \frac{\sin N\delta}{2\sqrt{N^3}} + \frac{\delta\cos N\delta}{\sqrt{N}}\right) \cdots ,$$

where $\mathcal{C}$ and $\mathcal{S}$ are the tabulated Fresnel integrals.[3] These are rapidly convergent for N large or δ small. Thus for $\delta = 0$, we have

$C = \Sigma n^{-1/2} \approx 2\sqrt{N} - 1.46 + 1/2\sqrt{N}$; for $N > 50$, this expression yields results agreeing to at least three places with those obtained by adding tabulated values of $n^{-1/2}$. We find $C = 18.6, 61.7, 198.5$ for $N = 10^2, 10^3, 10^4$, respectively.

For values of δ near π we write $\delta = \pi - \epsilon$ and rewrite (1) as

$$(3) \quad E(N,\delta) = \sum_{n=1}^{N} \frac{(-1)^n e^{-in\epsilon}}{\sqrt{n}} = -\sum_{n=1}^{N} \frac{e^{-in\epsilon}}{\sqrt{n}} + \sqrt{2}\sum_{n=1}^{N/2} \frac{e^{-in\epsilon/2}}{\sqrt{n}}$$

$$= -E(N,-\epsilon) + \sqrt{2}\,E(N/2 - \epsilon/2),$$

or as

$$(3') \quad E(N,\delta) = -C(N,\epsilon) + \sqrt{2}\,C(N/2, \epsilon/2) + i\left[S(N,\epsilon) - \sqrt{2}\,S(N/2, \epsilon/2)\right]$$

in order to use the representations of (2). Thus for $\delta = \pi$,

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[2] See, for example, L. Bieberbach, Lehrbuch der Funktionen Theorie (Chelsea Publishing Co., N.Y., 1945) Vol. I, p. 307.

[3] E. Jahnke and F. Emde , Tables of Functions (Dover Publications, N.Y., 1945) p. 35.

$$C = \sum_{n=1}^{N} (-1)^n n^{-1/2} = -\sum_{1}^{N} n^{-1/2} + \sqrt{2} \sum_{1}^{N/2} n^{-1/2} \approx -0.6 + 1/2\sqrt{N}.$$

Figures 25, 27, 29 are plots of S vs. C for $N = 10^2, 10^3, 10^4$ respectively and Figures 26, 28, 30 are the corresponding plots of C and S as functions of $u = (2N\delta/\pi)^{1/2}$. With increasing N, the 'axis' of the spiral tends to become parallel to the line C = S. Figures 31, 32 and 33 are enlargements of the tails of the spirals for $N = 10^2$, 10^3 and 10^4 respectively. Figure 34 is the analogous curve for $N = \infty$ obtained by Sollfrey and Shmoys.[4] It can be seen that in this range of δ (say $\delta = 40^\circ$ to 180°) the function is practically independent of N, provided that N is large.

The values of the graphs are accurate to about five percent.

To illustrate the use of the function $E(N,\delta)$ we consider the scattering of a plane electromagnetic wave normally incident on an infinite grating of parallel cylinders with radius a and spacing b. If we assume that the incident wave is polarized parallel to the cylinders, and that $Ka \ll 1$, $Kb \gg 1$ $(K = 2\pi/\lambda)$, we may write the total field (the amplitude of the electric intensity) as

$$(4) \quad \bar{\psi} = e^{iKx} + A \sum_{\nu=-\infty}^{\nu=+\infty} H_0^{(1)} \left(K\sqrt{x^2 + (y-\nu b)^2}\right), \quad \sqrt{x^2 + (y-\nu b)^2} \equiv r_\nu .$$

For perfectly conducting cylinders we impose the boundary condition $\psi = 0$ at $r_\nu = a$ and obtain

$$0 = 1 + A\left[H_0^{(1)}(Ka) + 2\sum_{n=1}^{\infty} H_0^{(1)}(nKb)\right];$$

we have employed $\exp(iKa\cos\Theta) \approx 1$, $\Theta = \tan^{-1}(y/x)$, for simplicity. Hence we may write

$$(5) \quad A = A_0/\left[1 - 2A_0 \sum H_0^{(1)}(nKb)\right], \quad A_0 = -1/H_0^{(1)}(Ka), \quad Ka \ll 1,$$

where A_0 is essentially the scattering coefficient of an isolated cylinder.

The scattered component of (4) is then[5]

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[4] W. Sollfrey and J. Shmoys, Diffraction of Electromagnetic Waves by a Plane Wire Grating, EM-42 (Math. Res. Grp., N.Y.U., N.Y., 1952).

[5] Solutions of this form were obtained previously by R. Honerjager, Ann. Physik 4, 25 (1948) and W. Franz, Z. ang. Phys. 9, 416 (1949). The present derivation is due to S. Karp and J.B. Keller.

$$(6) \quad \psi = \frac{A_0 \sum H_0^{(1)}(Kr_\nu)}{1 - 2A_0 \sum H_0^{(1)}(nKb)} = \frac{\psi_1}{D} .$$

The numerator ψ_1 is the result obtained if each cylinder is treated as completely isolated from the others, i.e., the 'single scattering' solution. The denominator D may be expanded as a power series and ψ readily interpreted physically in terms of higher order scattering processes; ψ is thus an approximate 'multiple scattering' solution for the problem.

Since $Kb >> 1$ we may employ the asymptotic form of $H_0^{(1)}(nKb)$ to obtain

$$(7) \quad \psi / \psi' = D^{-1} = [1 - 2A_0 (2/\pi Kb)^{1/2} e^{-i\pi/4} E(N, \delta)]^{-1}$$

where N is the number of neighbors we take into account, and $\delta = Kb - 2m\pi$ is the phase difference between waves from successive neighbors. Graphs of $|D|^{-2}$ and additional illustrations will be found in reference [1].

Figures 1 to 10

Graphs of S vs. C for N = 1 to 10; the parameter δ ranges from 0° to 180°, the curves being symmetrical to the real axis.

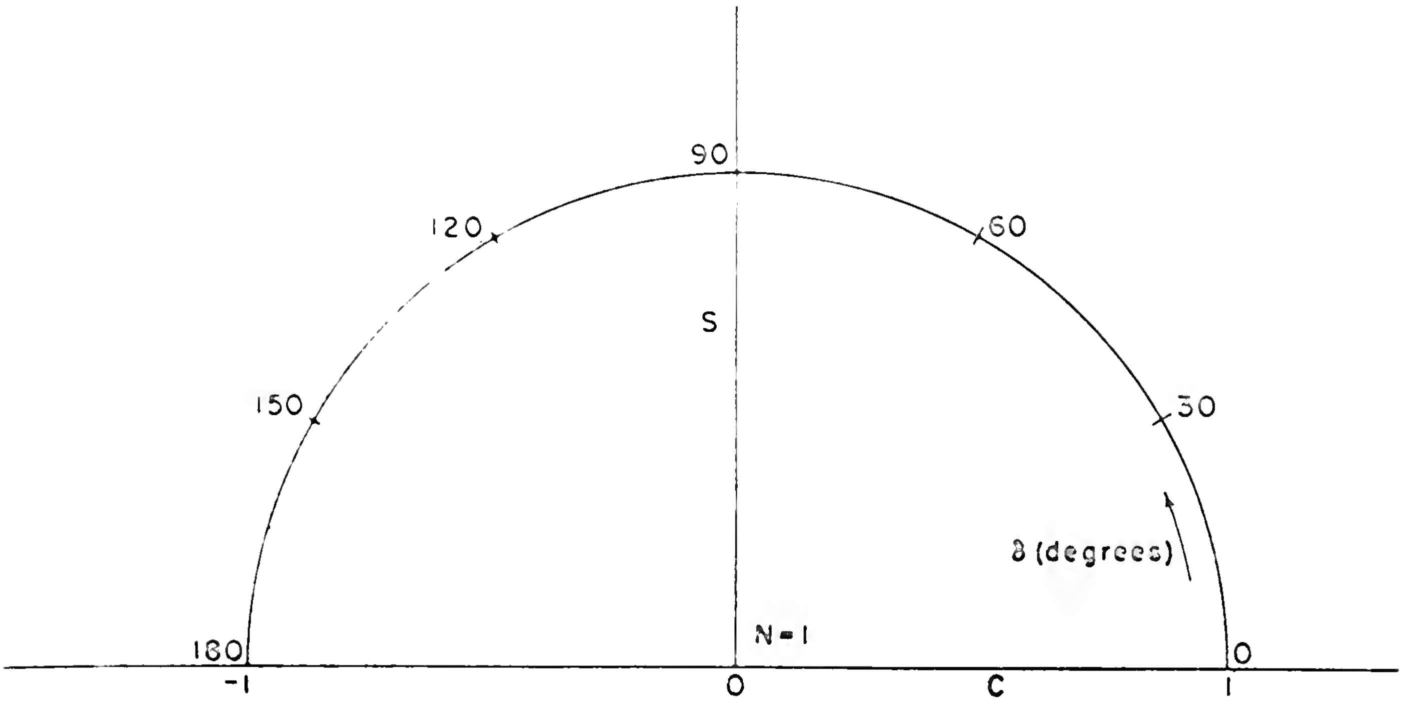
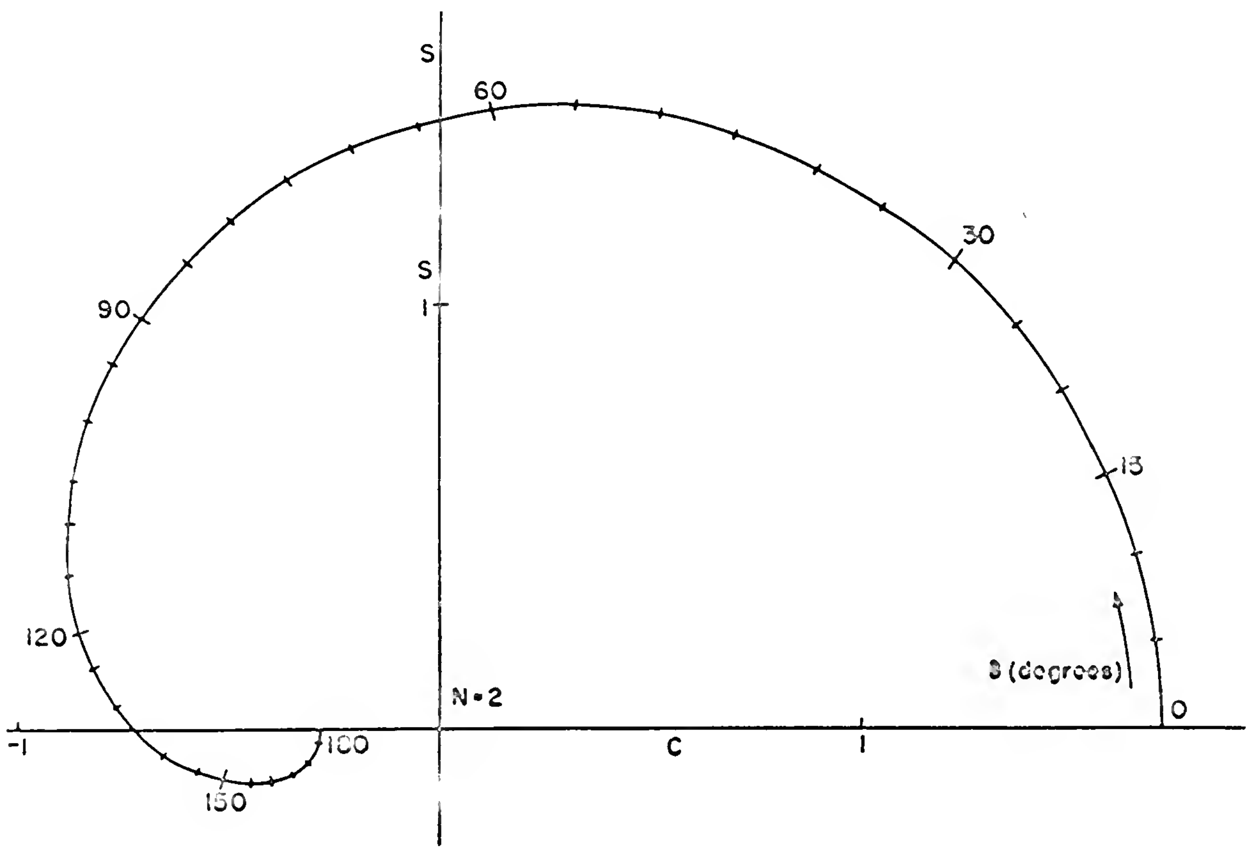


Figure 1



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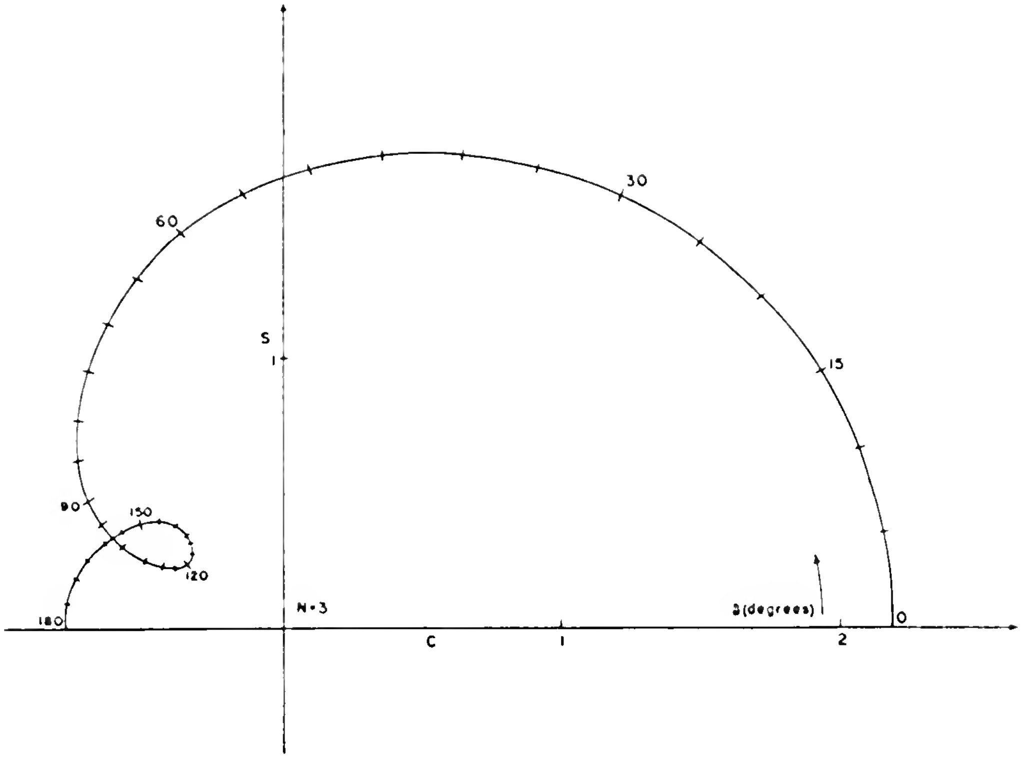

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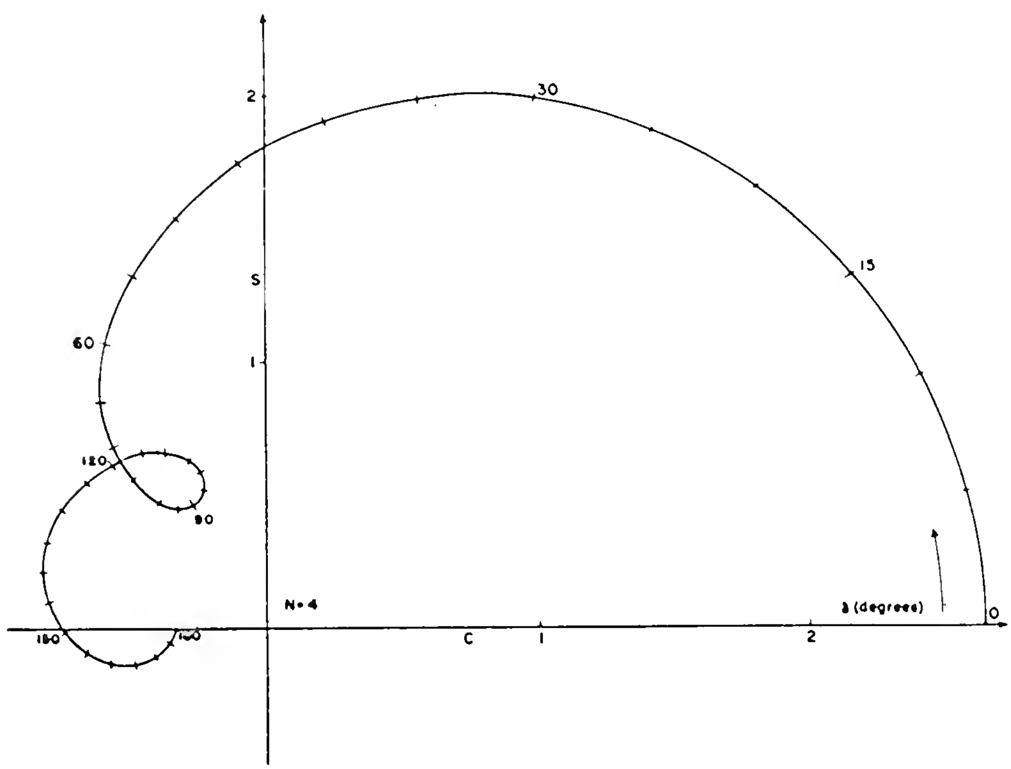

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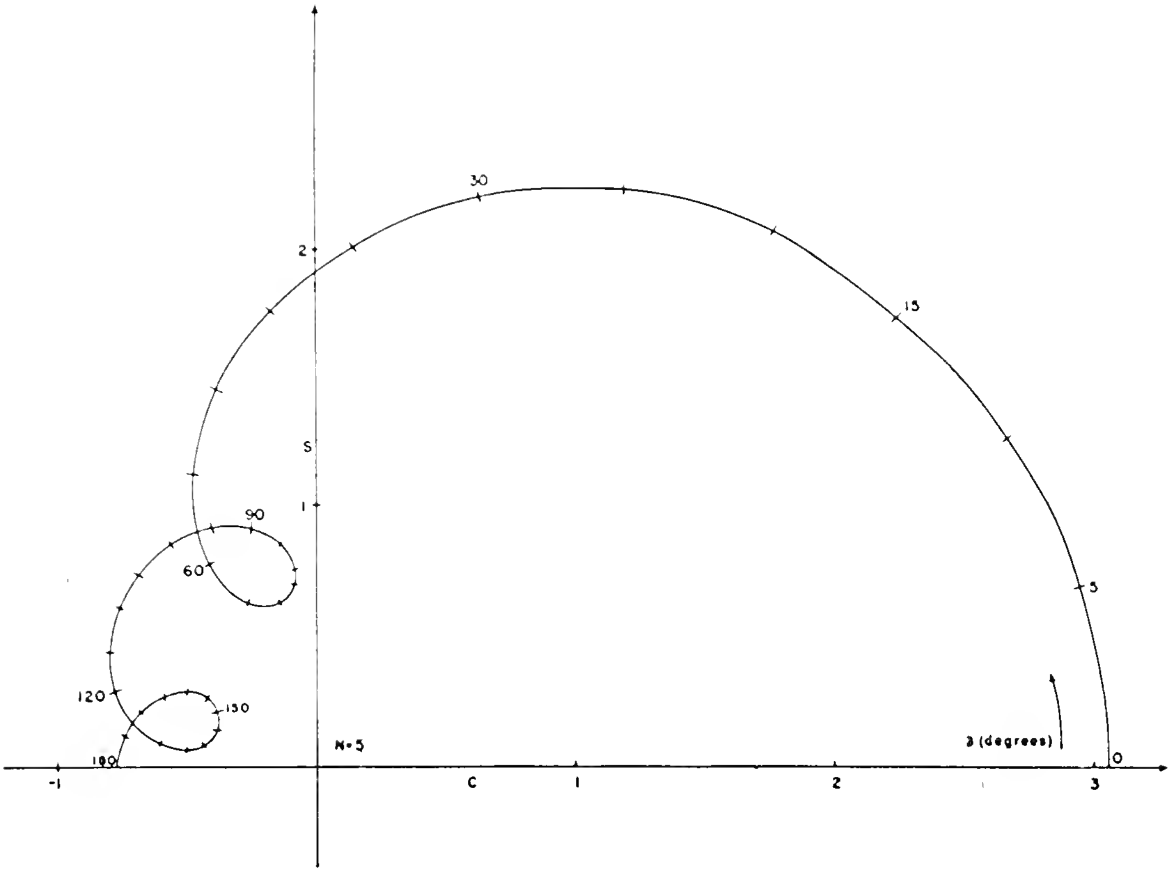

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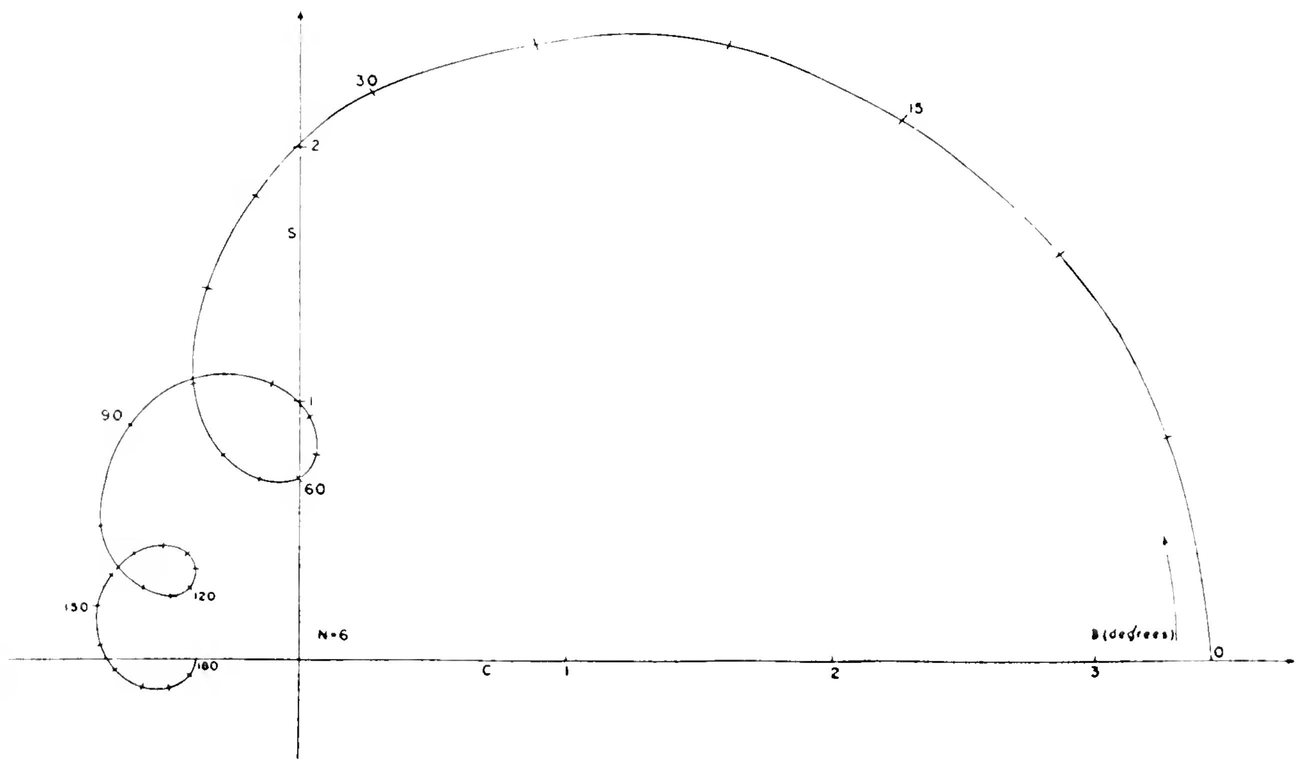

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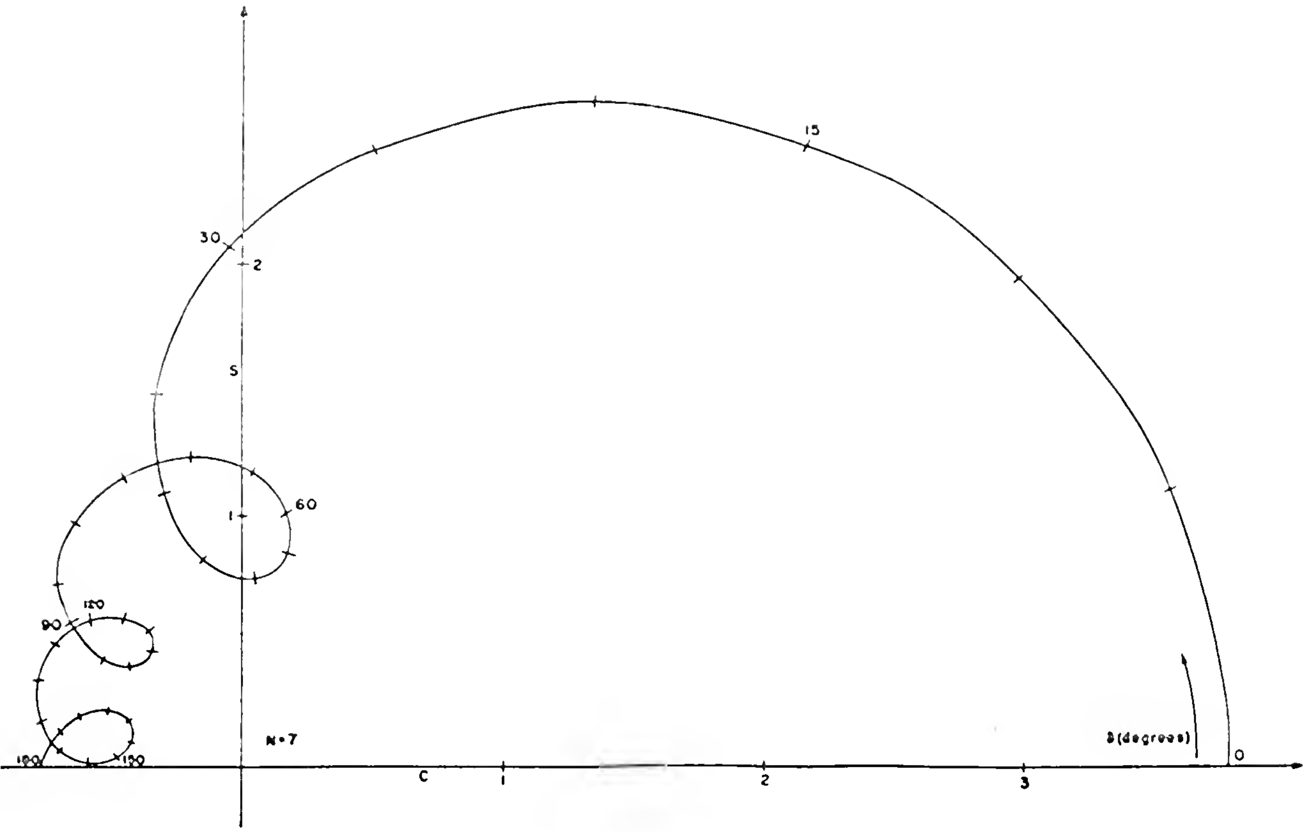

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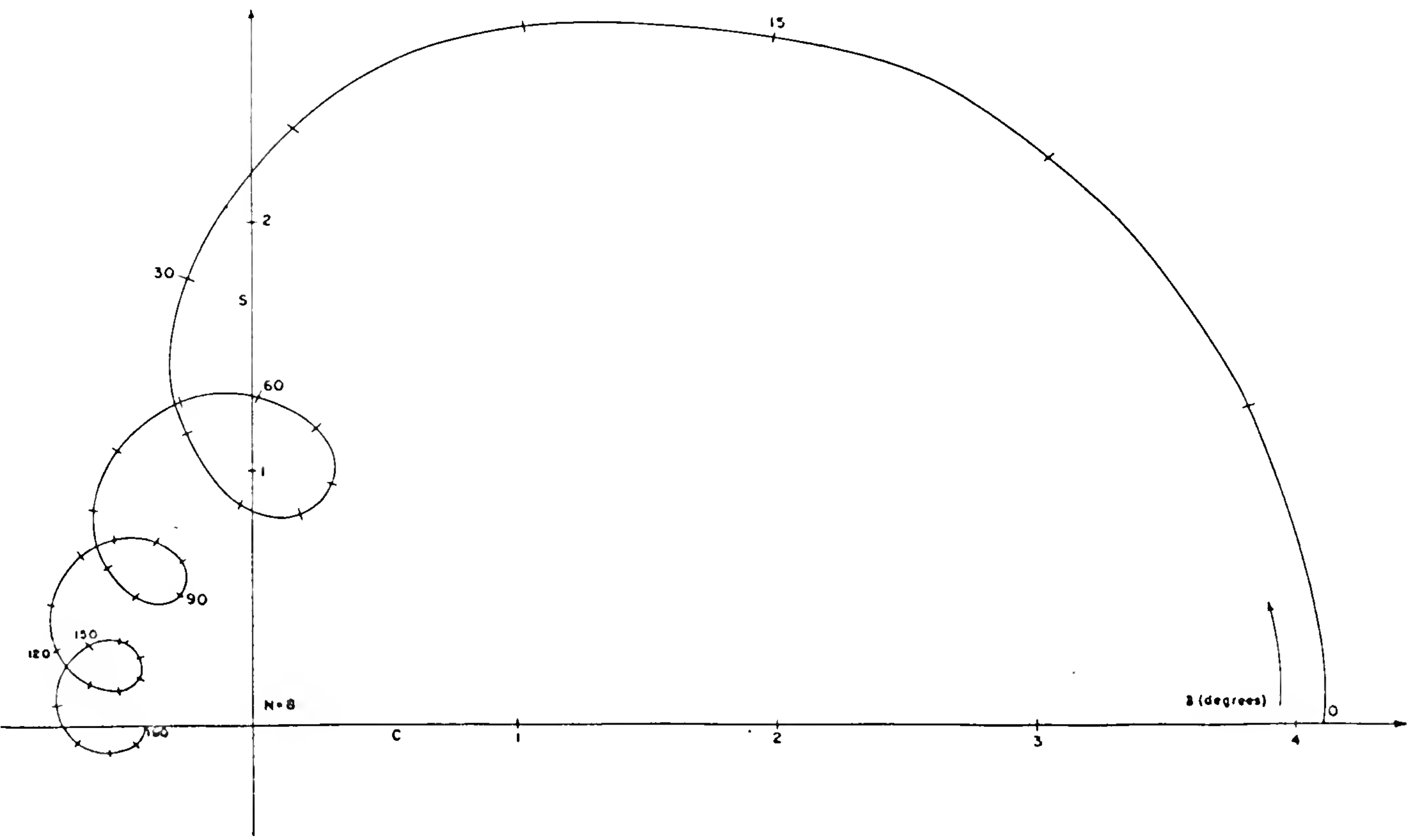

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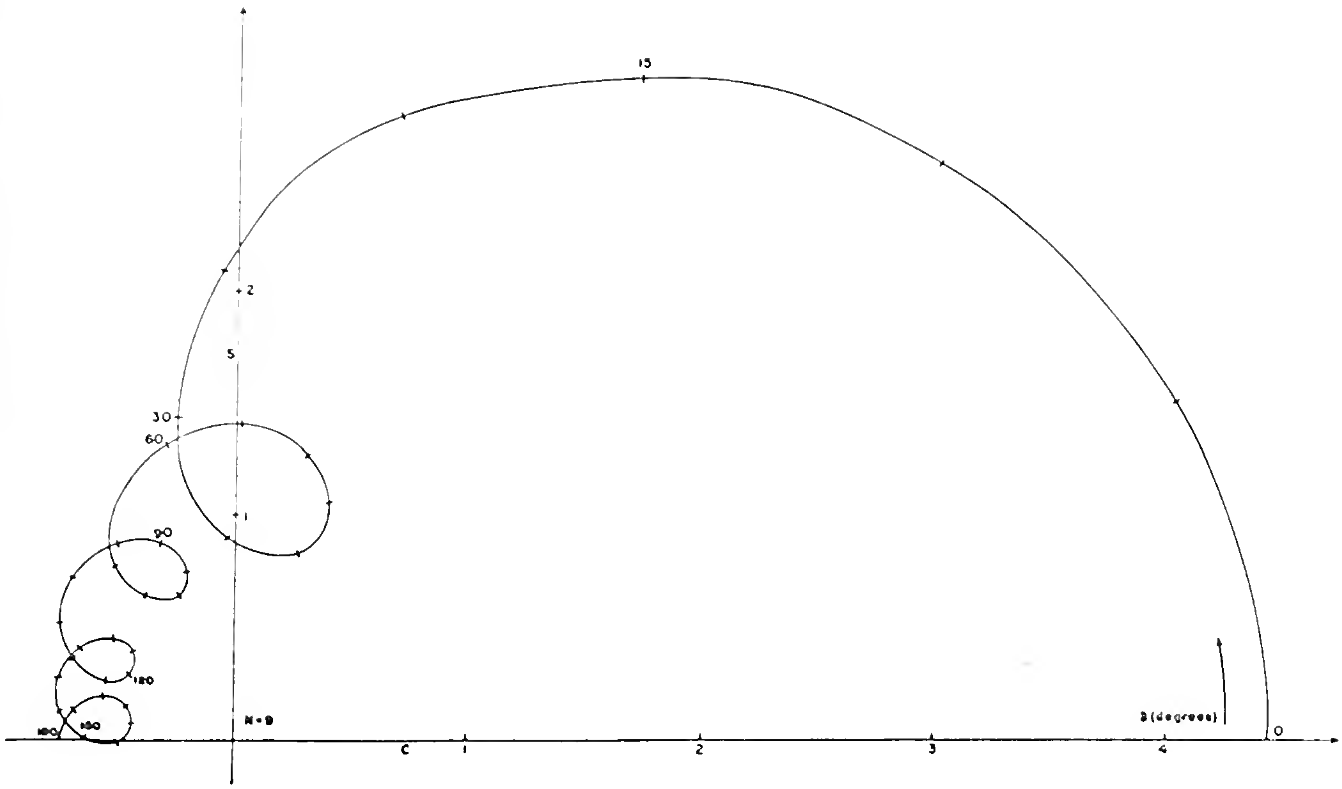

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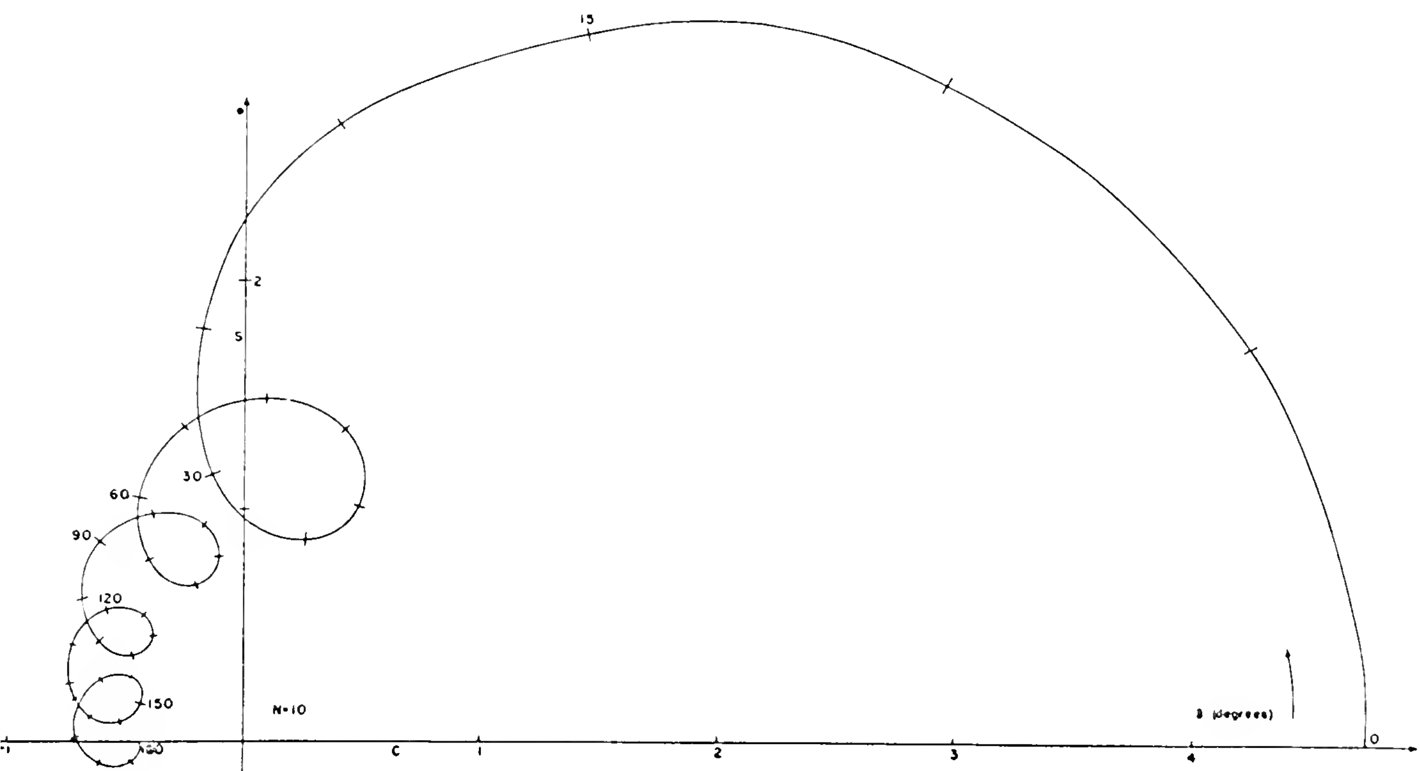
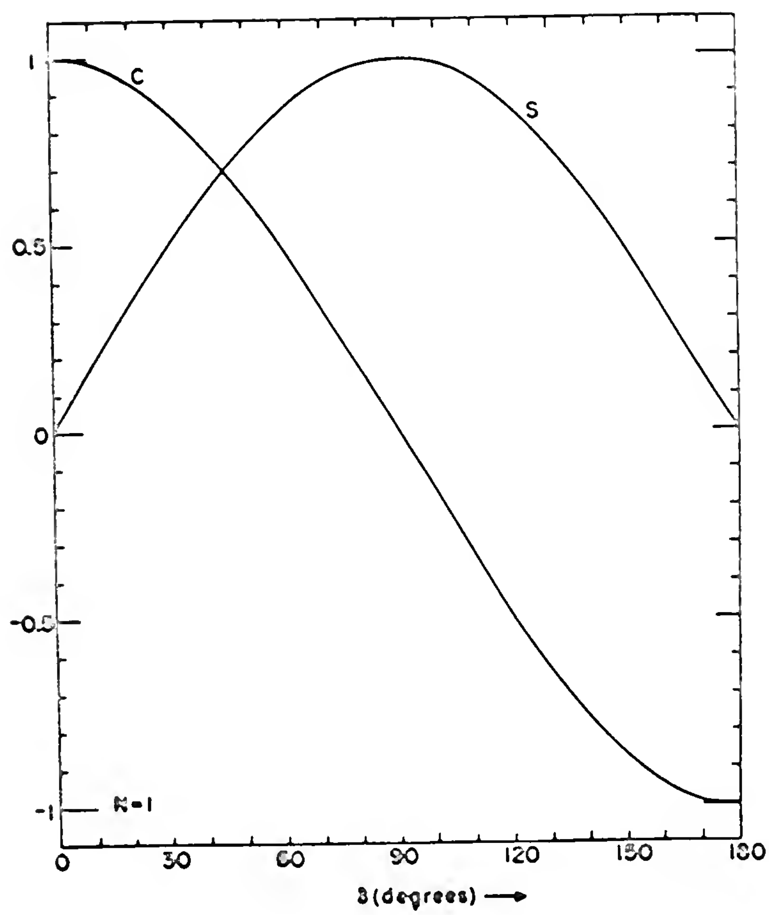


Figure 10

Figures 11 to 20

C and S as functions of δ for N = 1 to 10



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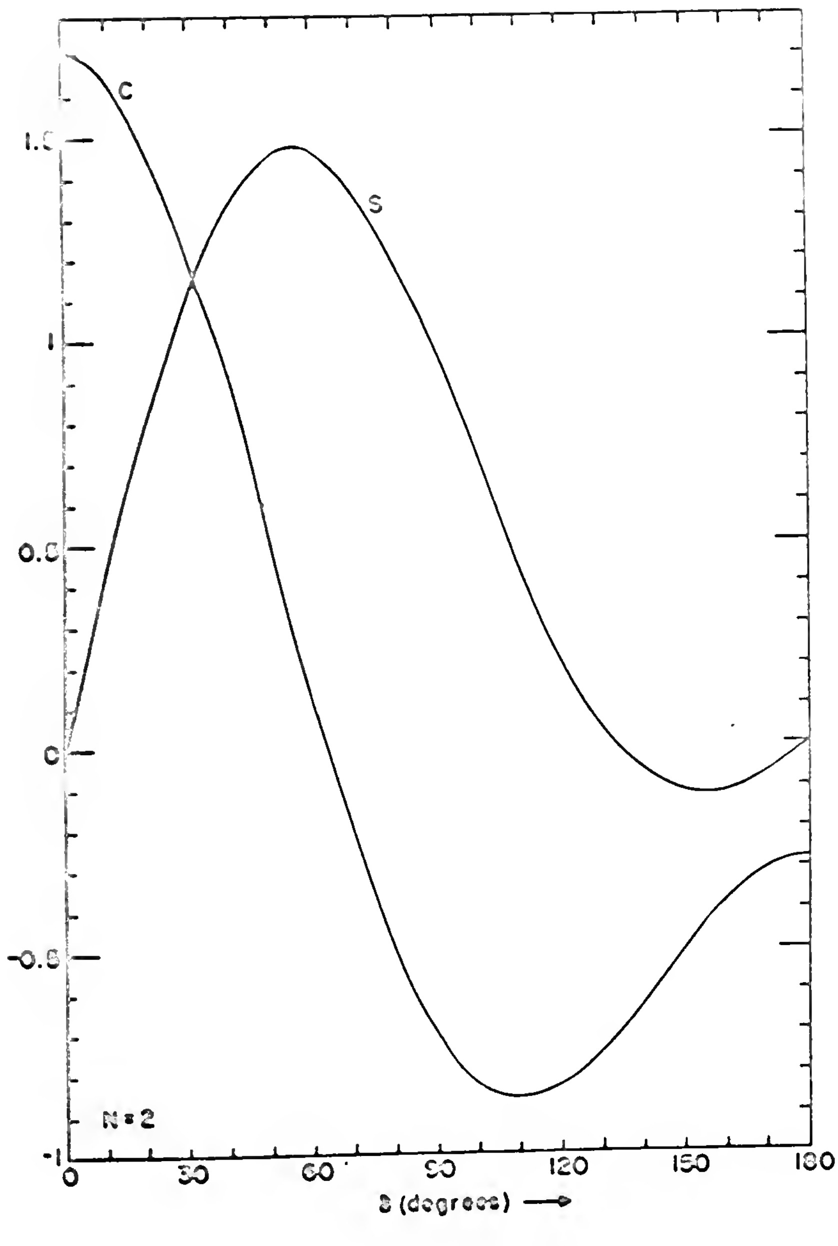

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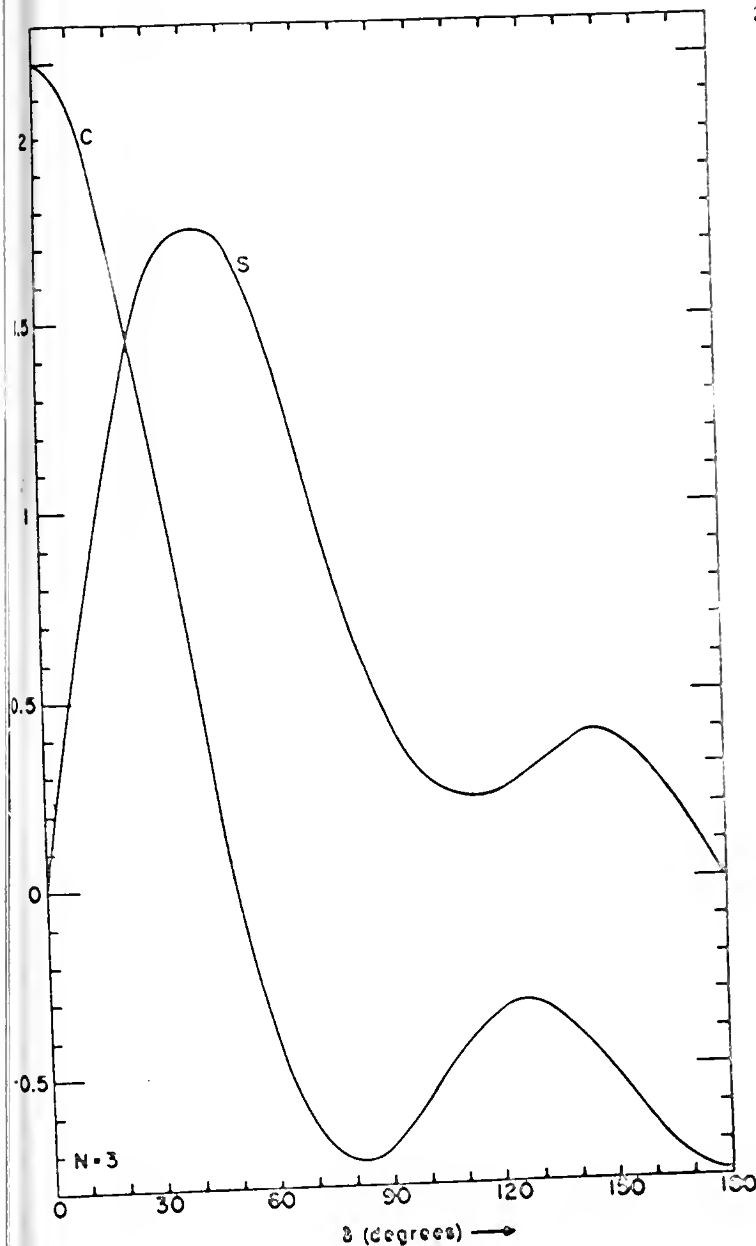

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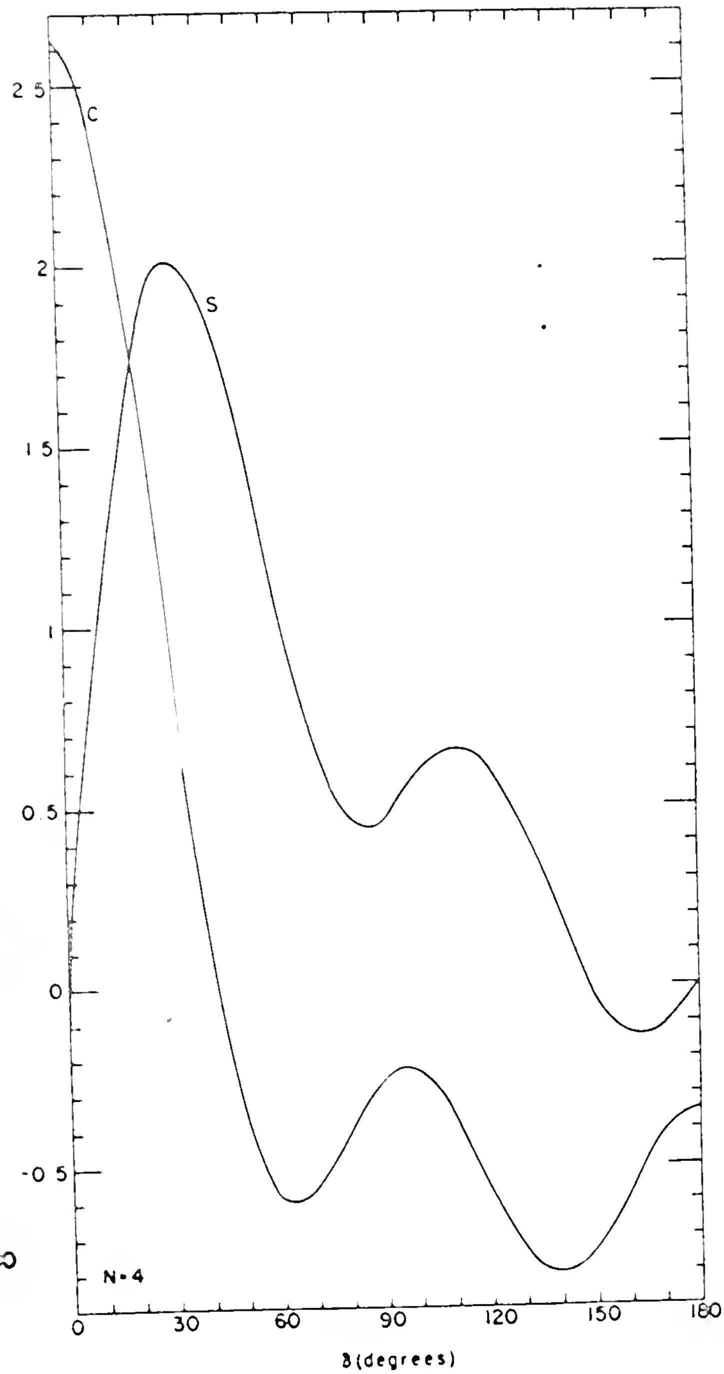

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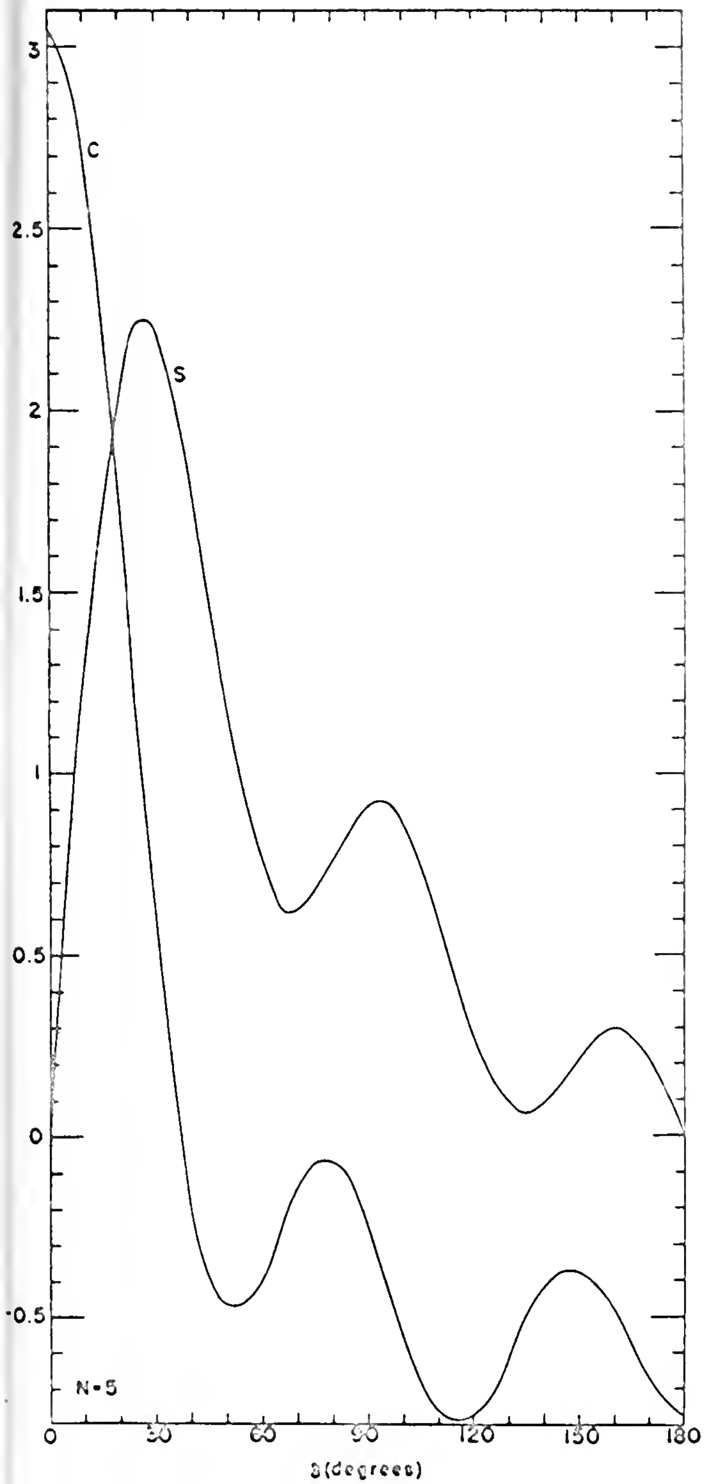

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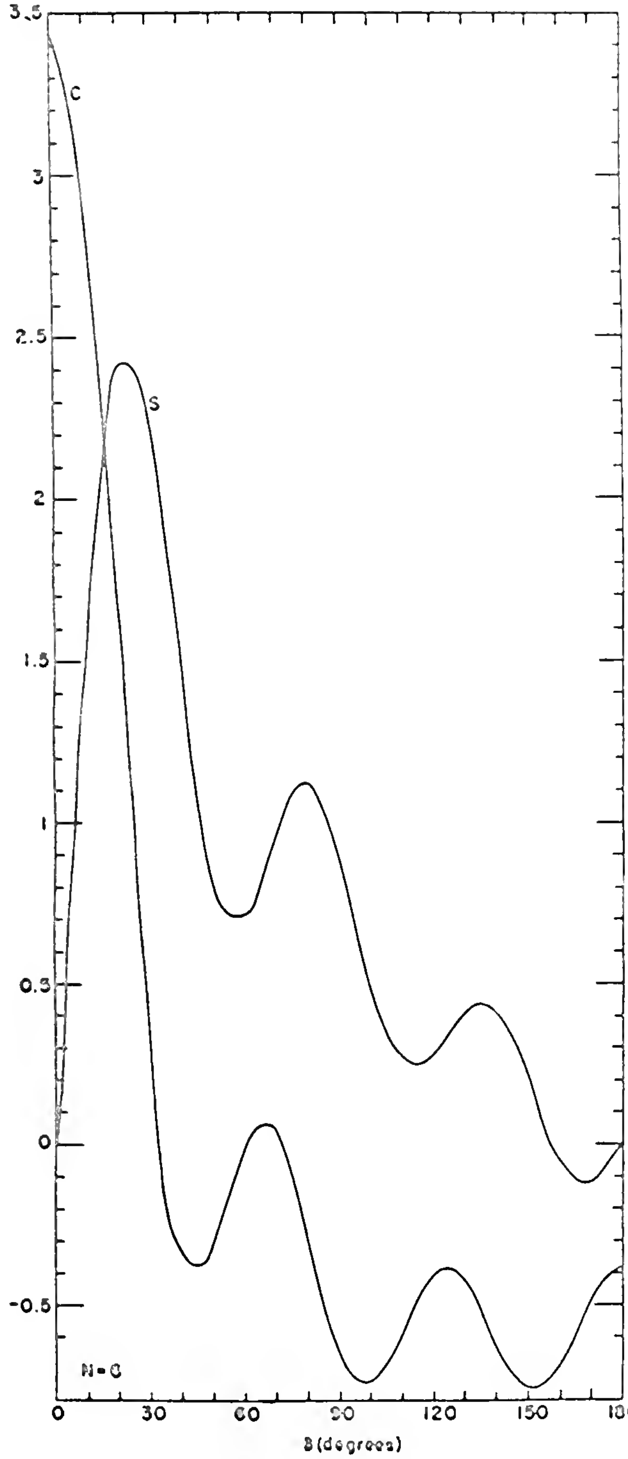

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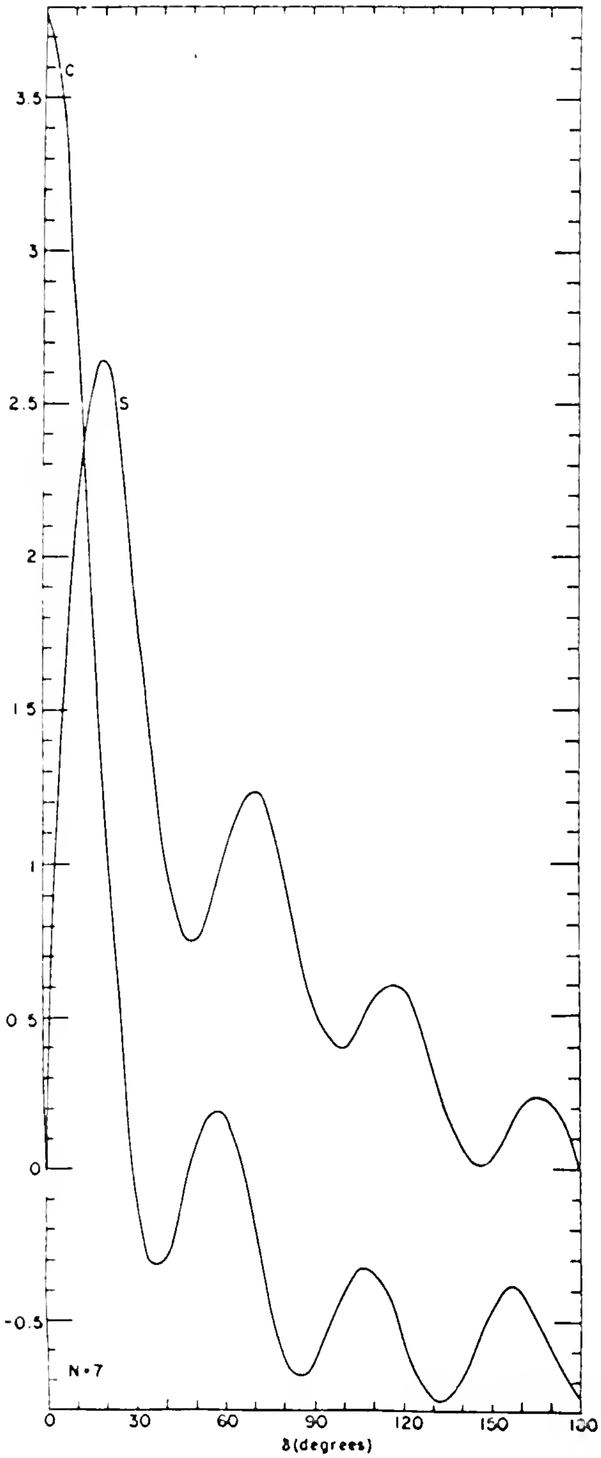

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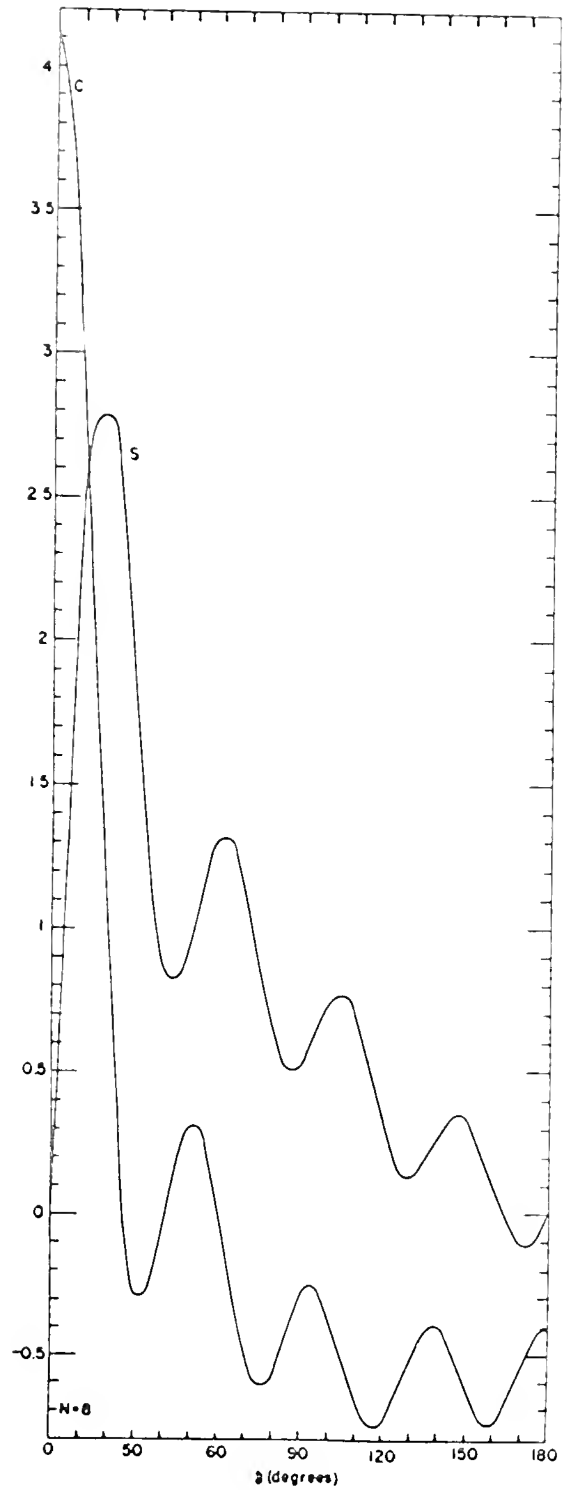

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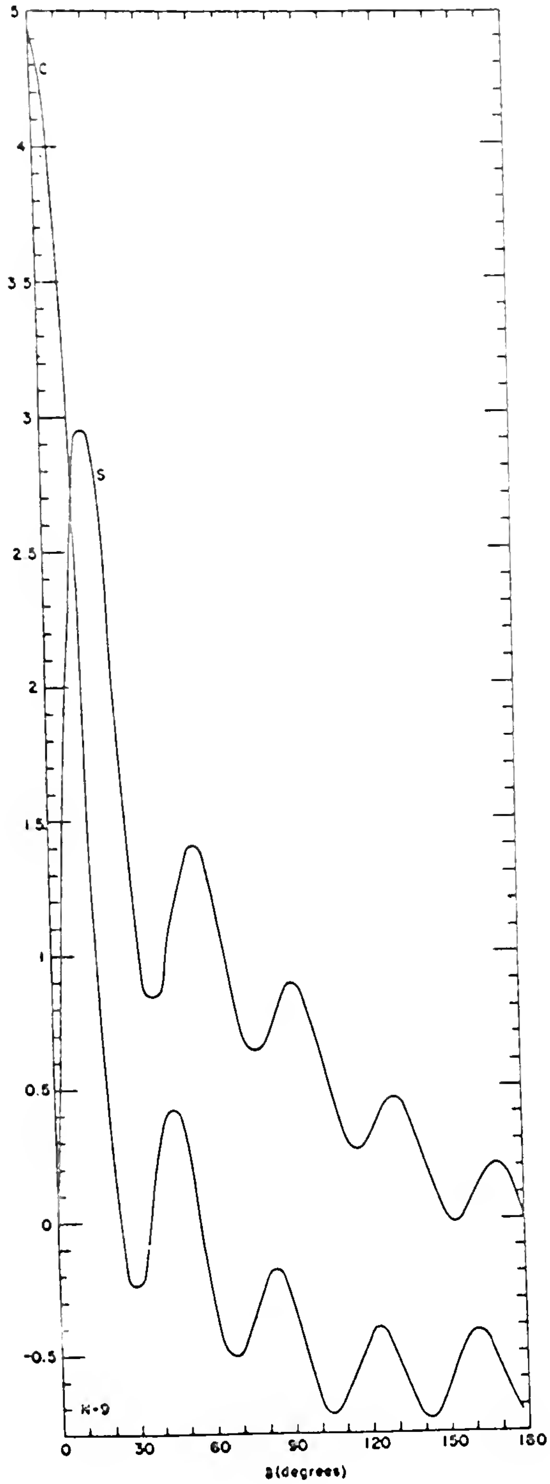

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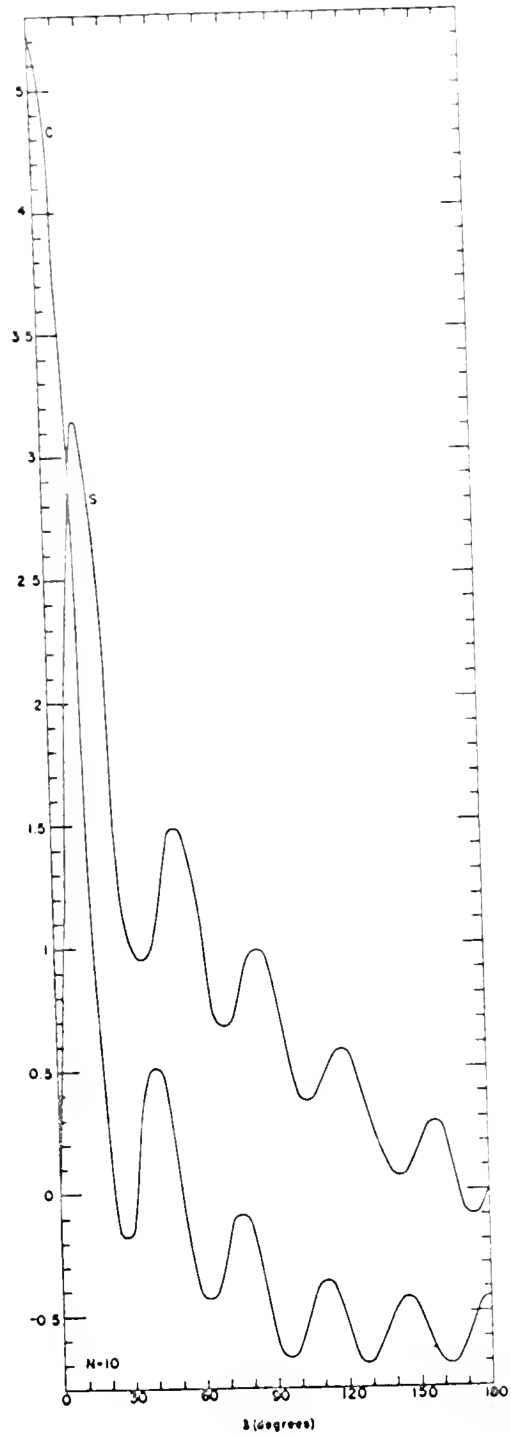
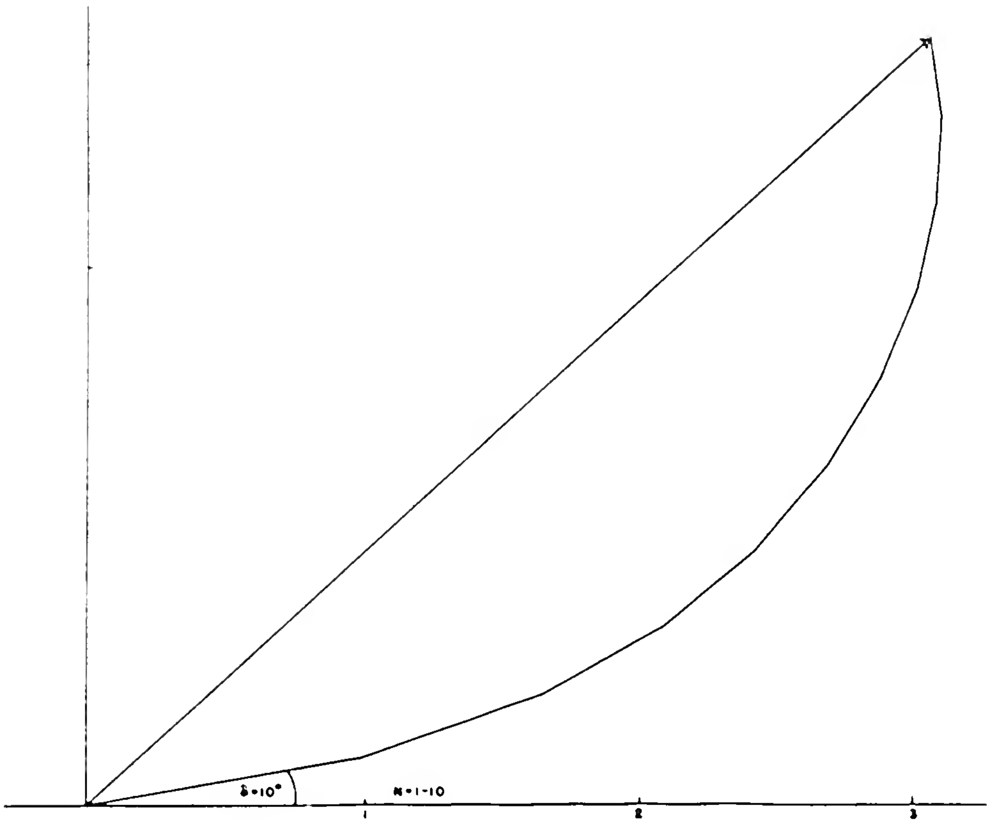


Figure 20

Figures 21 to 24

Vector plots for δ constant and N increasing from 1 to 10



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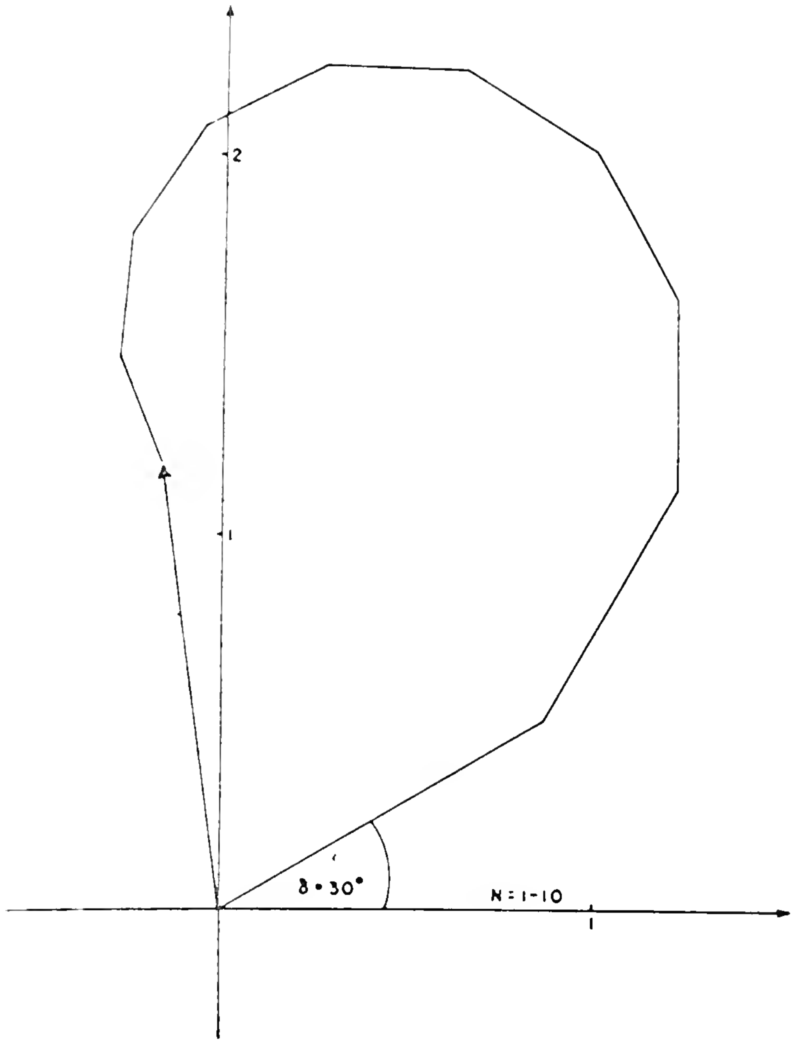

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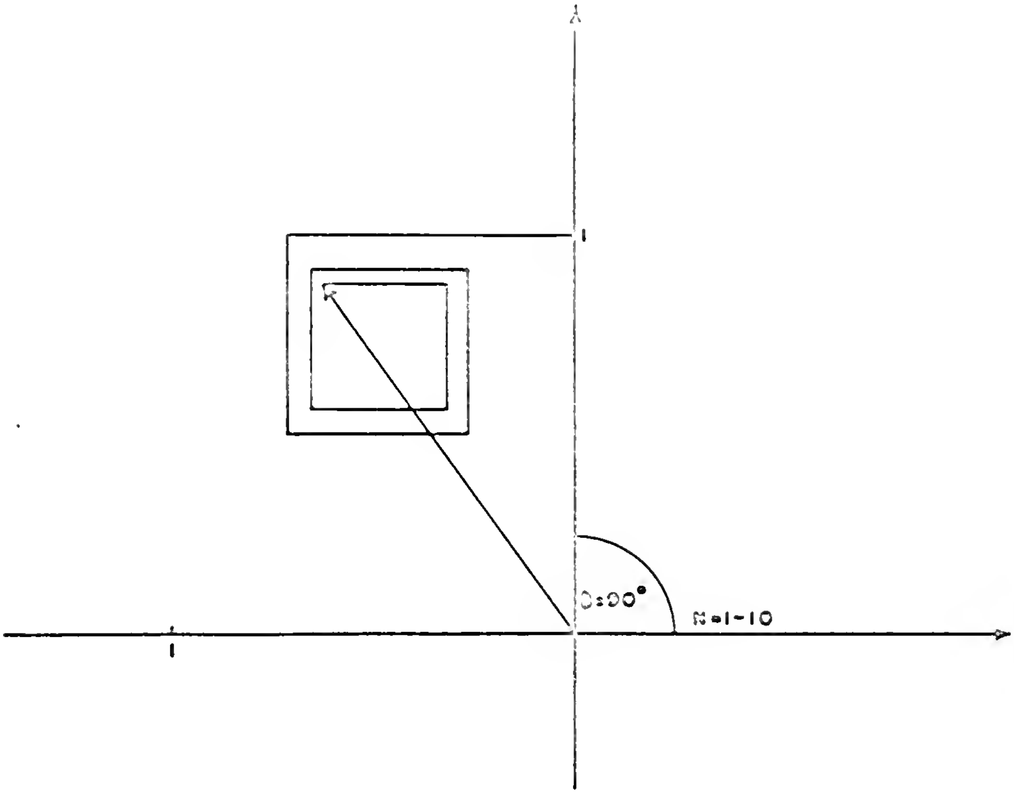

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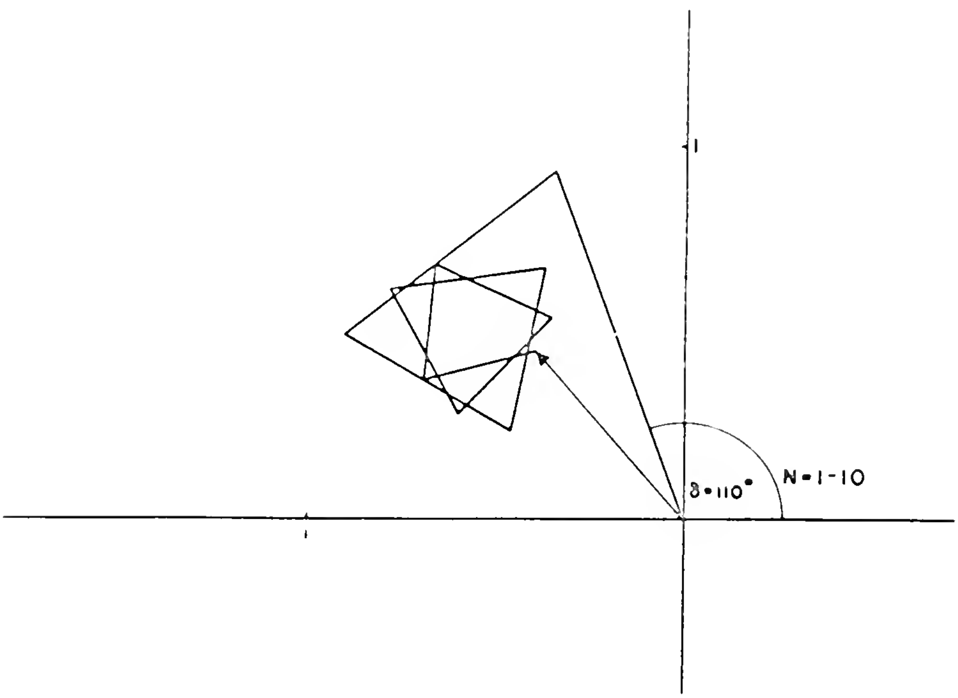

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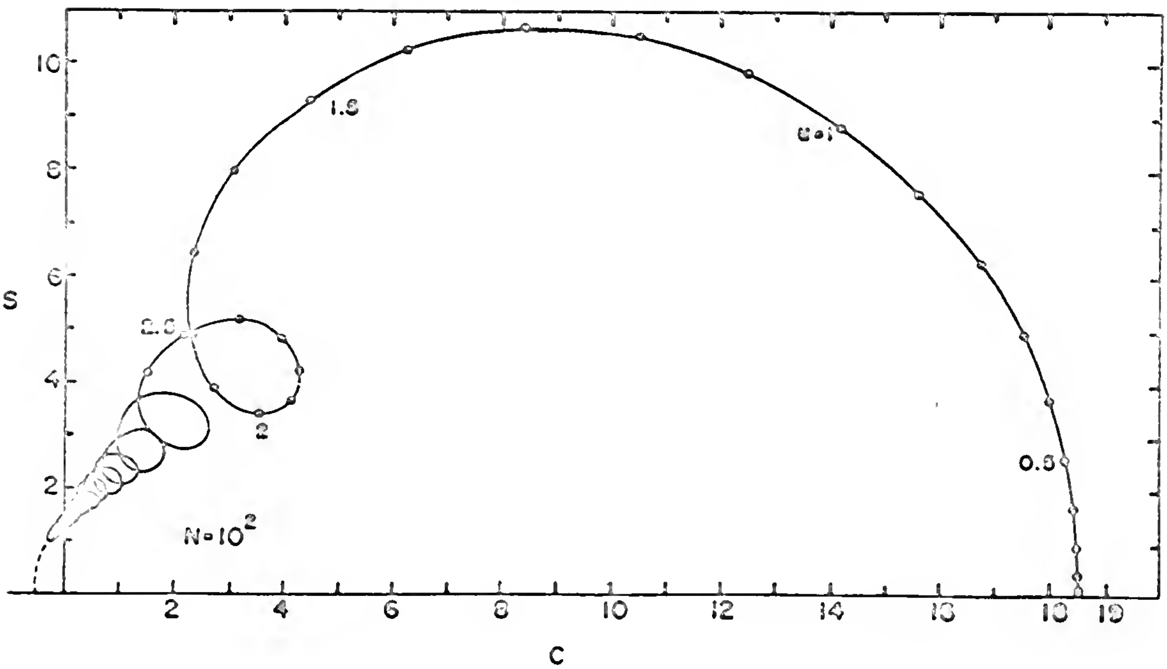


Figure 25. C vs. S as functions of u for $N = 10^2$

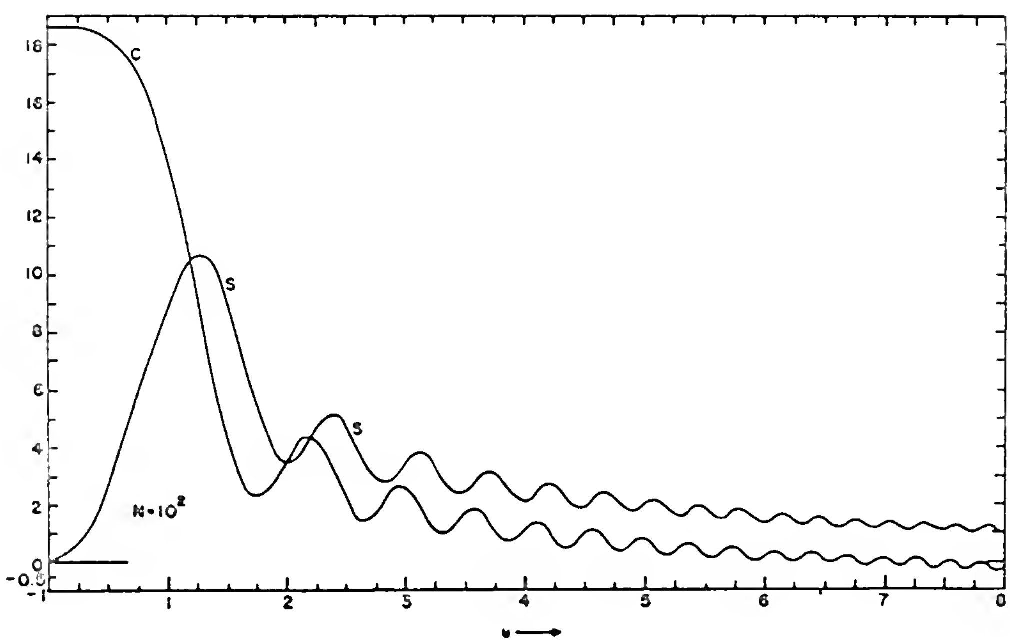


Figure 26. C and S as functions of u for $N = 10^2$

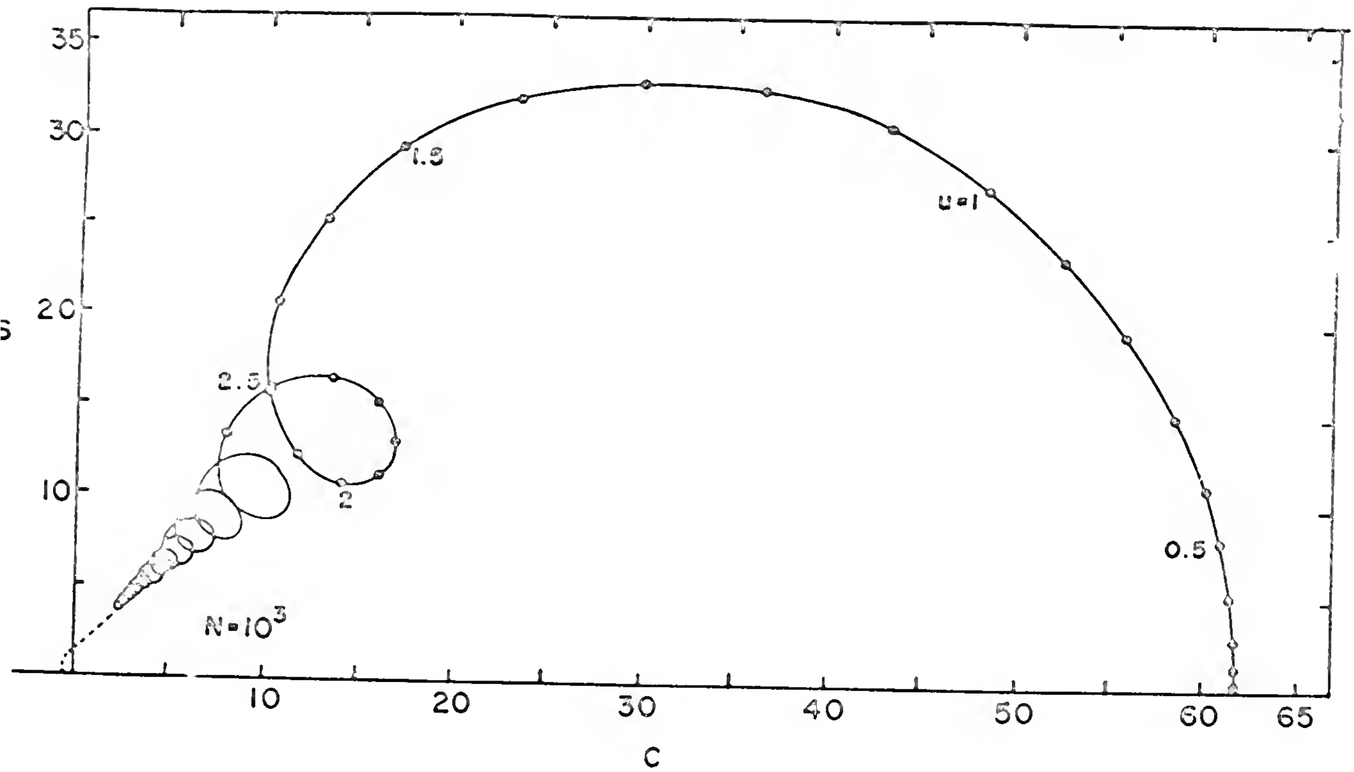


Figure 27. C vs. S as functions of u for $N = 10^3$

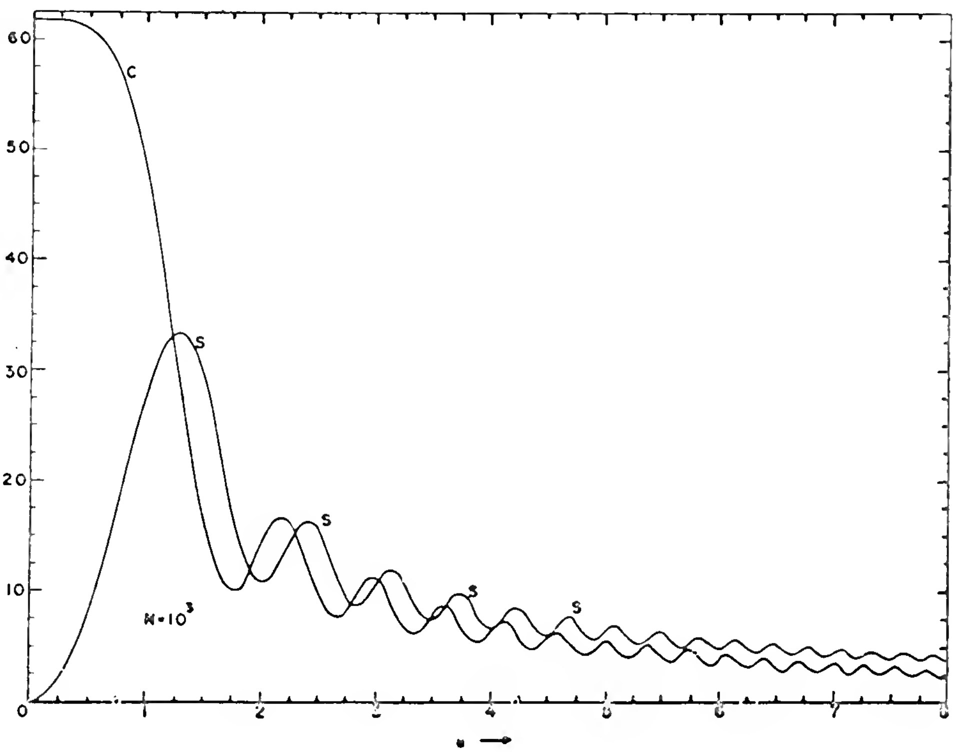


Figure 28. C and S as functions of u for $N = 10^3$

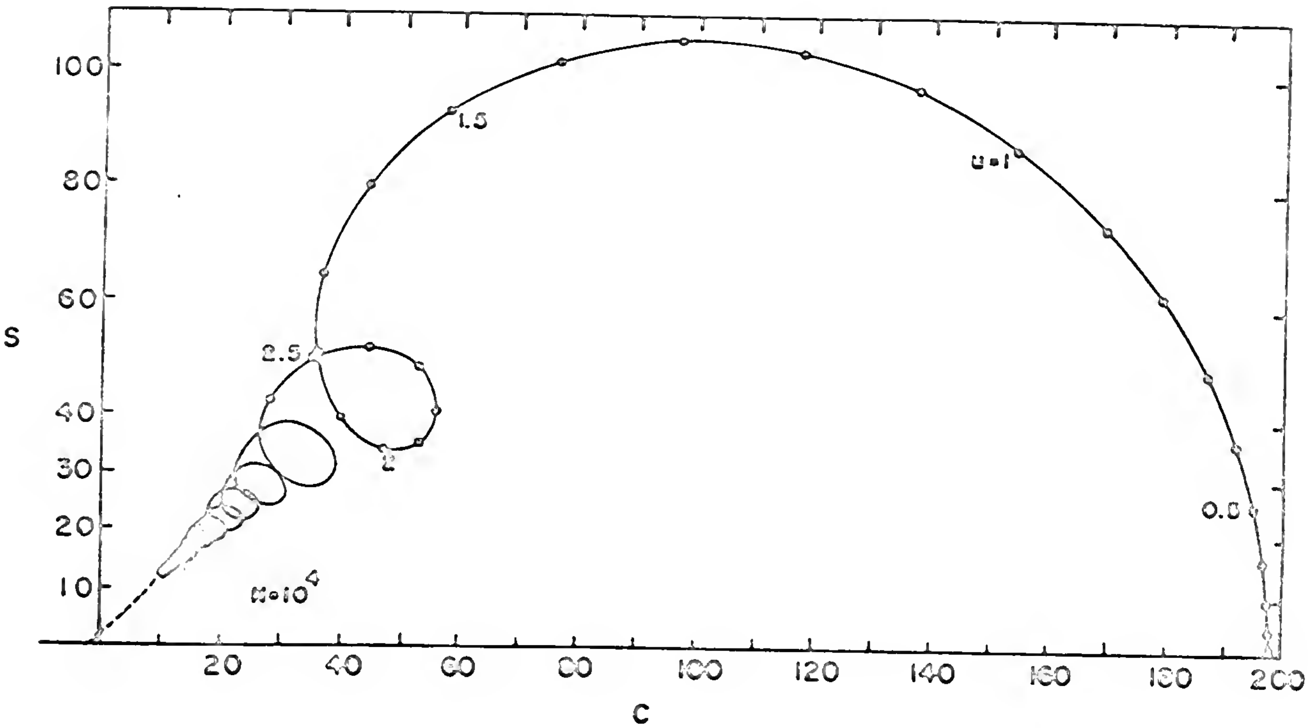


Figure 29. C vs. S as functions of u for N = 10^4

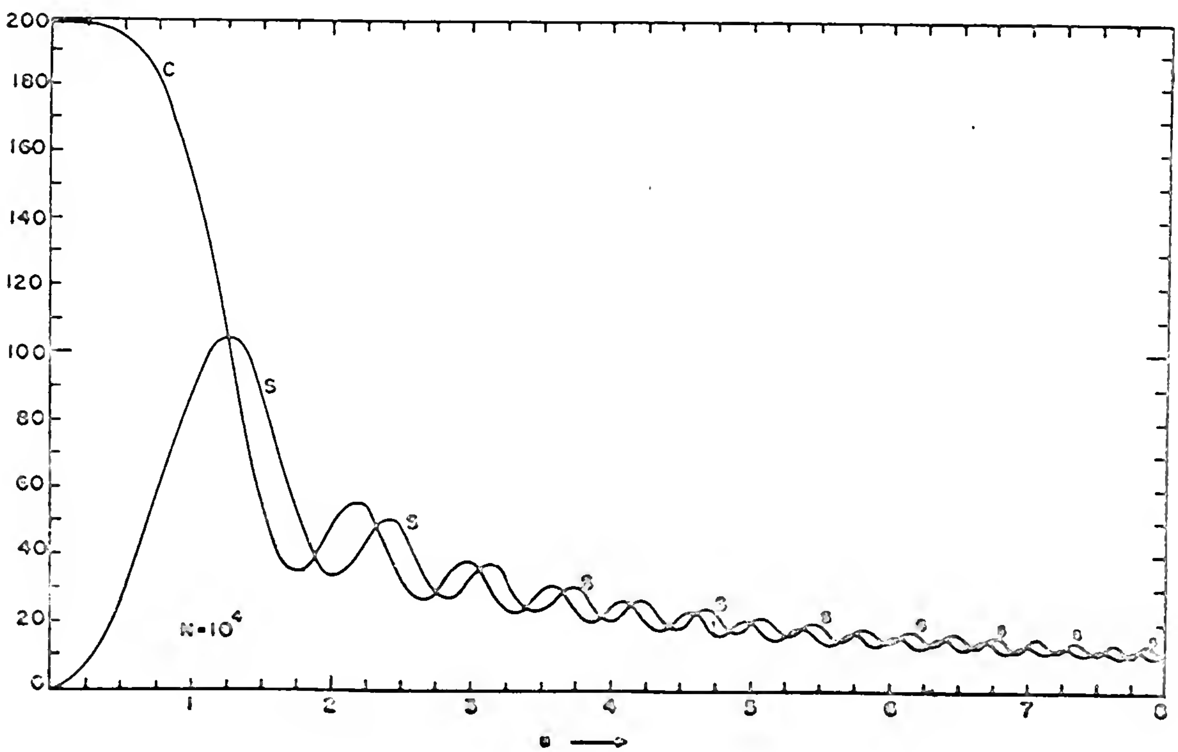
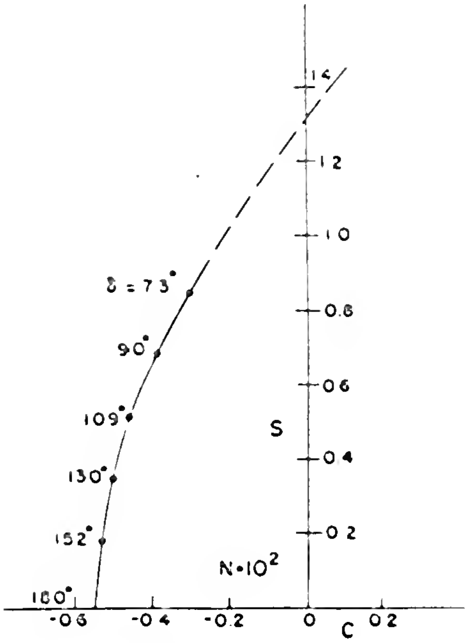


Figure 30. C and S as functions of u for N = 10^4

Figures 31 to 34

Enlargements of the C vs. S curves for the vicinity of $\delta = \pi$



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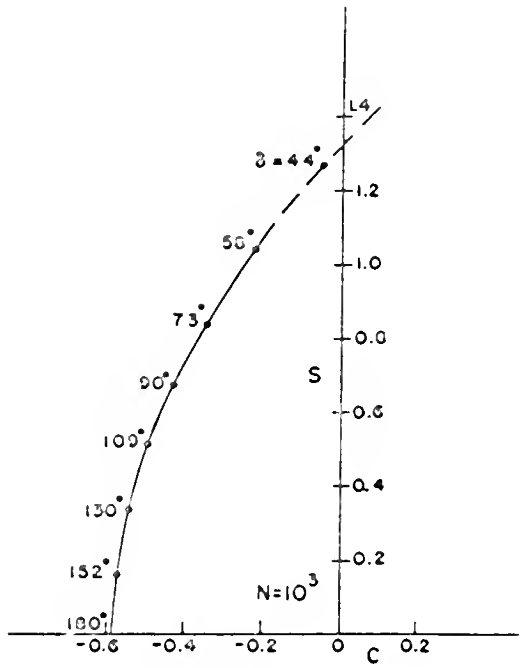

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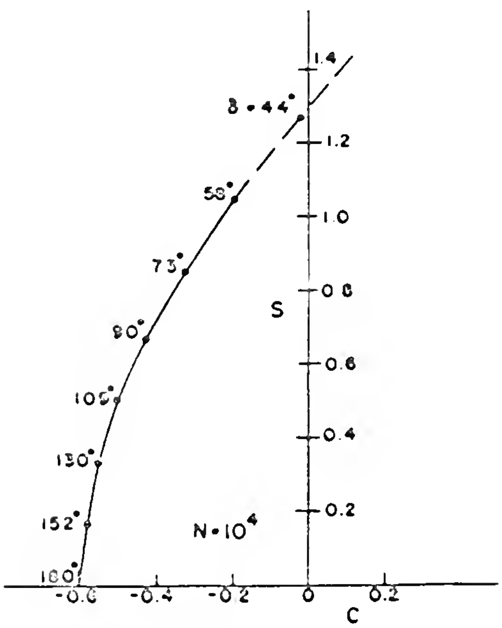

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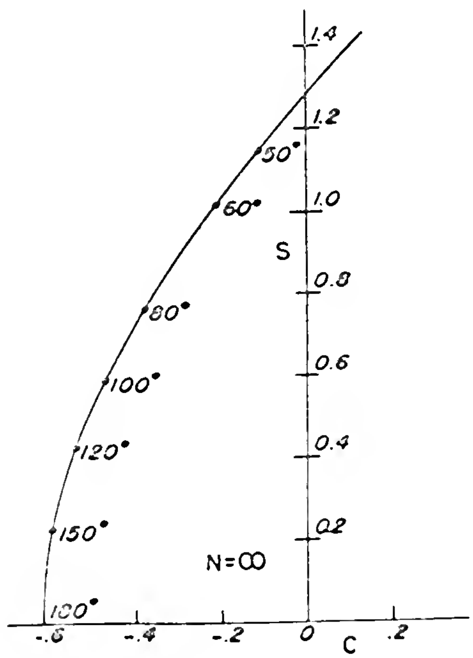

Figure 34